Potential theory of Markov processes with jump kernels decaying at the boundary

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Reference

This talk is based on the following paper:

[KSV1] P. Kim, R. Song and Z. Vondracek: On potential theory of Markov processes with jump kernels decaying at the boundary. **Pot. Anal. 58** (2023), 465–528

[KSV2] P. Kim, R. Song and Z. Vondracek: Sharp two-sided Green function estimates for Dirichlet forms degenerate at the boundary. To appear in **J. European Math. Soc.**

[KSV3] P. Kim, R. Song and Z. Vondracek: Potential theory of Dirichlet forms degenerate at the boundary: the case of no killing potential. To appear in **Math. Ann.**

[CKSV] S. Cho, P. Kim, R. Song and Z. Vondracek: Heat kernel estimates for Dirichlet forms degenerate at the boundary. arXiv:2211.08606

Outline

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In the last few decades, a lot of progress has been made in the study of potential theoretic properties for various types of jump processes in open subsets D of $\mathbb{R}^d$. These include killed symmetric Lévy processes, and their censored and reflected versions.

An important example of a killed symmetric Lévy process is the killed symmetric α-stable process X^D in $D \subset \mathbb{R}^d$. X^D can also be constructed from Brownian motion as follows. Suppose B is a Brownian motion in $\mathbb{R}^d$, S is an independent $\alpha/2$ -stable subordinator. Then the process $X_t := B_{S_t}$ is a symmetric α-stable process. Kill this process when it exits D, we get a killed symmetric α-stable process X^D. Here we do subordination first and then killing. X^D is a killed subordinate Brownian motion. If we replace the Brownian motion by a general symmetric Lévy process and S by a general subordinator, we get a killed subordinate Lévy process.

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In these studies, the jump kernel $J^D(x, y)$ of the process in the open set D is either the restriction of the jump kernel of the original process in $\mathbb{R}^d$ or comparable to such a kernel, and it does not tend to zero as x or y tends to the boundary of D. For example, the jump kernels of the killed stable process, and censored stable processes in D are both $c|x-y|^{-d-\alpha}$.

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Suppose X is a symmetric Lévy process in $\mathbb{R}^d$, D is an open subset of $\mathbb{R}^d$, and S_t is an independent subordinator. The process $Y_t := X^D_{S_t}$ is called a subordinate killed Lévy process. Here we perform killing first and then subordination. These two operations are not communtative, a subordinate killed Lévy process is different from a killed subordinate Lévy process.

In case X is a Brownian motion, S_t is an $\alpha/2$-stable subordinator, Y is a subordinate killed BM and its generator is $-(-\Delta|_D)^{\alpha/2}$, called the spectral fractional Laplacian in the PDE community. In case X is a symmetric β-stable process, S_t is an independent $\gamma/2$-subordinator, Y is a subordinate killed β-stable process and its generator is $-((-\Delta)^{\beta/2}|_D)^{\gamma/2}$.

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Subordinate killed Lévy processes form another important class of Markov processes. Unlike killed Lévy processes and censored processes, the jump kernel of a subordinate killed Lévy process in an open subset $D \subset \mathbb{R}^d$ tends to zero near the boundary of D. In this sense, the Dirichlet forms of subordinate killed Lévy processes are degenerate near the boundary.

In two earlier papers, Kim-S.-Vondraček (**TAMS**, 2019) and Kim-S.-Vondraček (**Pot. Anal.**, 2019), we studied the potential theory of those processes. There were some unexpected results.

(i) The boundary Harnack principle holds in certain cases and fails in other cases. (For $-((-\Delta)^{\beta/2}|_D)^{\gamma/2}$ with $\beta \in (0,2)$, BHP holds when $\gamma > 1$, fails when $\gamma \in (0,1]$.)

(ii) There is a phase transition in the estimates of the jump kernel.

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Subordinate killed Lévy processes are natural and important, but their structures are too rigid for applications. In some sense we are just dealing with particular processes. Is there is a general theory behind all these?

Using the results of Kim-S.-Vondracek (**TAMS**, 2019) and Kim-S.-Vondraček (**Pot. Anal.**, 2019) as guidelines, in ([KSV1], we built a general framework to study the potential theory of Markov processes with jump kernels decaying at the boundary.

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Outline

Let $\mathbb{R}^d_+ = \{x = (\widetilde{x}, x_d) : \ x_d > 0\}$, $j(|x-y|) = |x-y|^{-\alpha-d}$, $0 < \alpha < 2$. Let $\mathcal{B}(x,y)$ be a function on $\mathbb{R}^d_+ \times \mathbb{R}^d_+$ satisfying the following assumptions:

(A1) $\mathcal{B}(x,y) = \mathcal{B}(y,x)$ for all $x, y \in \mathbb{R}^d_+$.

(A2) If $\alpha \geq 1$, then there exist $\theta > \alpha - 1$ and $C_1 > 0$ such that

$$|\mathcal{B}(x,x) - \mathcal{B}(x,y)| \leq C_1 \left(\frac{|x-y|}{x_d \wedge y_d} \right)^{\theta}.$$

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(A3) There exist $C_2 \geq 1$ and parameters $\beta_1, \beta_2, \beta_3 \geq 0$, with $\beta_1 > 0$ if $\beta_3 > 0$, and $\beta_2 > 0$ if $\beta_4 > 0$, such that

$$C_2^{-1} B_{\beta_1,\beta_2,\beta_3,\beta_4}(x,y) \leq \mathcal{B}(x,y) \leq C_2 B_{\beta_1,\beta_2,\beta_3,\beta_4}(x,y)\,, \qquad x, y \in \mathbb{R}^d_+\,,$$

where $B_{\beta_1,\beta_2,\beta_3,\beta_4}(x,y)$ is defined to be

$$\left(\frac{x_d \wedge y_d}{|x-y|} \wedge 1\right)^{\beta_1} \left(\frac{x_d \vee y_d}{|x-y|} \wedge 1\right)^{\beta_2} \left[\log\left(1 + \frac{(x_d \vee y_d) \wedge |x-y|}{x_d \wedge y_d \wedge |x-y|}\right)\right]^{\beta_3} \times$$
$$\times \left[\log\left(1 + \frac{|x-y|}{(x_d \vee y_d) \wedge |x-y|}\right)\right]^{\beta_4}.$$

(A4) For all $x, y \in \mathbb{R}^d_+$ and $a > 0$, $\mathcal{B}(ax, ay) = \mathcal{B}(x,y)$. In case $d \geq 2$, for all $x, y \in \mathbb{R}^d_+$ and $\widetilde{z} \in \mathbb{R}^{d-1}$, $\mathcal{B}(x + (\widetilde{z}, 0), y + (\widetilde{z}, 0)) = \mathcal{B}(x,y)$.

Define

$$J(x,y) = |x-y|^{-d-\alpha}\mathcal{B}(x,y), \qquad x, y \in \mathbb{R}^d_+ \times \mathbb{R}^d_+.$$

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$$J(x,y) = |x-y|^{-d-\alpha}\mathcal{B}(x,y), \qquad x,y \in \mathbb{R}^d_+ \times \mathbb{R}^d_+.$$

Let $\mathbf{e}_d = (\tilde{0}, 1)$. To every parameter $p \in ((\alpha - 1)_+, \alpha + \beta_1)$, we associate a constant $C(p) = C(\alpha, p, \mathcal{B}) \in (0, \infty)$ defined as

$$C(p) = \int_{\mathbb{R}^{d-1}} \frac{1}{(|\widetilde{u}|^2 + 1)^{(d+\alpha)/2}} \int_0^1 \frac{(s^p - 1)(1 - s^{\alpha-p-1})}{(1-s)^{1+\alpha}} \mathcal{B}((1-s)\widetilde{u}, 1), s\mathbf{e}_d)\, ds d\widetilde{u}\,,$$

In case $d = 1$, $C(p)$ is defined as

$$C(p) = \int_0^1 \frac{(s^p - 1)(1 - s^{\alpha-p-1})}{(1-s)^{1+\alpha}} \mathcal{B}(1, s)\, ds.$$

Note that $\lim_{p \downarrow (\alpha-1)_+} C(p) = 0$, $\lim_{p \uparrow \alpha+\beta_1} C(p) = \infty$ and that the function $p \mapsto C(p)$ is strictly increasing.

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Let $\kappa(x) = \kappa x_d^{-\alpha}$ on $\mathbb{R}^d_+$.

Define

$$\mathcal{E}^\kappa(u,v) := \frac{1}{2}\int_{\mathbb{R}^d_+}\int_{\mathbb{R}^d_+}(u(x)-u(y))(v(x)-v(y))J(x,y)\,dy\,dx + \int_{\mathbb{R}^d_+} u(x)v(x)\kappa(x)dx.$$

Let $\mathcal{F}^\kappa$ be the closure of $C_c^\infty(\mathbb{R}^d_+)$ under $\mathcal{E}_1^\kappa(u,u) = \mathcal{E}^\kappa(u,u) + (u,u)$. Then $(\mathcal{E}^\kappa, \mathcal{F}^\kappa)$ is a Dirichlet form, degenerate at the boundary due to **(A3)**.

Let $((Y_t^\kappa)_{t\geq 0}, (\mathbb{P}_x)_{x\in\mathbb{R}^d_+})$ be the associated Hunt process with lifetime ζ^κ. We add a cemetery point ∂ to the state space $\mathbb{R}^d_+$ and define $Y_t^\kappa = \partial$ for $t \geq \zeta^\kappa$.

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The generator of the process Y^κ is

$$L^{\mathcal{B}} f(x) = \text{p.v.} \int_{\mathbb{R}^d_+} (f(y) - f(x)) J(x,y)\, dy - \kappa x_d^{-\alpha} f(x), \quad x \in \mathbb{R}^d_+.$$

Let $p \in ((\alpha-1)_+, \alpha+\beta_1)$ be such that $\kappa = C(p)$. If $g_p(x) = x_d^p$, then $L^{\mathcal{B}} g_p \equiv 0$. Hence the operator $L^{\mathcal{B}}$ annihilates the p-th power of the distance to the boundary.

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Outline

For $d \geq 2$ and $\widetilde{w} \in \mathbb{R}^{d-1}$, we define $D_{\widetilde{w}}(a, b)$ to be the box $\{x = (\widetilde{x}, x_d) \in \mathbb{R}^d_+ : |\widetilde{x} - \widetilde{w}| < a, x_d < b\}$. When $d = 1$, we will use $D_{\widetilde{w}}(a, b)$ to stand for the open interval $(0, b) = \{y \in \mathbb{R}_+ : 0 < y < b\}$.

Theorem 1(Boundary Harnack principle) [KSV1, KSV2]

Suppose $p \in ((\alpha - 1)_+, \alpha + (\beta_1 \wedge \beta_2))$. Then there exists $C \geq 1$ such that for all $r > 0$, $\widetilde{w} \in \mathbb{R}^{d-1}$, and any non-negative function f in $\mathbb{R}^d_+$ which is harmonic in $D_{\widetilde{w}}(2r, 2r)$ with respect to Y^κ and vanishes continuously on $B((\widetilde{w}, 0), 2r) \cap \partial\mathbb{R}^d_+$, we have

$$\frac{f(x)}{x_d^p} \leq C_3 \frac{f(y)}{y_d^p}, \quad x, y \in D_{\widetilde{w}}(r/2, r/2).$$

Theorem 2 [KSV1, KSV2]

If $d \geq 2$ and $\alpha + \beta_2 \leq p < \alpha + \beta_1$, then the boundary Harnack principle is not valid for Y^κ.

For $d \geq 2$ and $\widetilde{w} \in \mathbb{R}^{d-1}$, we define $D_{\widetilde{w}}(a,b)$ to be the box $\{x = (\widetilde{x}, x_d) \in \mathbb{R}^d_+ : |\widetilde{x} - \widetilde{w}| < a, x_d < b\}$. When $d = 1$, we will use $D_{\widetilde{w}}(a,b)$ to stand for the open interval $(0,b) = \{y \in \mathbb{R}_+ : 0 < y < b\}$.

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If $d \geq 2$ and $\alpha + \beta_2 \leq p < \alpha + \beta_1$, then the boundary Harnack principle is not valid for Y^κ.

Theorem 3 [KSV2]

Suppose $d > (\alpha + \beta_1 + \beta_2) \wedge 2$ and $p \in ((\alpha - 1)_+, \alpha + \beta_1)$. Let G^κ be the Green function of Y^κ. (1) If $p \in ((\alpha - 1)_+, \alpha + \frac{1}{2}[\beta_1 + (\beta_1 \wedge \beta_2)])$, then on $\mathbb{R}^d_+ \times \mathbb{R}^d_+$,

$$G^\kappa(x, y) \asymp \frac{1}{|x - y|^{d-\alpha}} \left(\frac{x_d}{|x - y|} \wedge 1 \right)^p \left(\frac{y_d}{|x - y|} \wedge 1 \right)^p . \qquad (1)$$

Theorem 3 (cont) [KSV2]

(2) If $p = \alpha + \frac{\beta_1+\beta_2}{2}$, then on $\mathbb{R}^d_+ \times \mathbb{R}^d_+$,

$$G^\kappa(x,y) \asymp \frac{1}{|x-y|^{d-\alpha}} \left(\frac{x_d}{|x-y|} \wedge 1 \right)^p \left(\frac{y_d}{|x-y|} \wedge 1 \right)^p \times \\ \times \left(\log \left(1 + \frac{|x-y|}{(x_d \vee y_d) \wedge |x-y|} \right) \right)^{\beta_4+1}.$$

(3) If $p \in (\alpha + \frac{\beta_1+\beta_2}{2}, \alpha+\beta_1)$, then on $\mathbb{R}^d_+ \times \mathbb{R}^d_+$,

$$G^\kappa(x,y) \asymp \frac{1}{|x-y|^{d-\alpha}} \left(\frac{x_d \wedge y_d}{|x-y|} \wedge 1 \right)^p \left(\frac{x_d \vee y_d}{|x-y|} \wedge 1 \right)^{2\alpha-p+\beta_1+\beta_2} \times \\ \times \left(\log \left(1 + \frac{|x-y|}{(x_d \vee y_d) \wedge |x-y|} \right) \right)^{\beta_4}.$$

In proving the three results above, the strict positivity of the killing function was used in a crucial way in several places.

What happens if the killing function is identically zero?

In [KSV3], we studied this case. For the next two results, we assume the killing function is identically zero. It is easy to show that when $\alpha \in (0, 1]$, the process Y^0 will not approach $\partial \mathbb{R}^d_+$ at the end of its lifetime, so there is no "boundary theory". So in this section, we also assume $\alpha \in (1, 2)$.

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Theorem 4 (Boundary Harnack principle), KSV3

There exists $C \ge 1$ such that for all $r > 0$, $\widetilde{w} \in \mathbb{R}^{d-1}$, and any non-negative function f in $\mathbb{R}^d_+$ which is harmonic in $D_{\widetilde{w}}(2r, 2r)$ with respect to Y^0 and vanishes continuously on $B((\widetilde{w}, 0), 2r) \cap \partial\mathbb{R}^d_+$, we have

$$\frac{f(x)}{x_d^{\alpha-1}} \le C\frac{f(y)}{y_d^{\alpha-1}}, \quad x, y \in D_{\widetilde{w}}(r/2, r/2).$$

Theorem 5 [KSV3]

Then there exists $C > 1$ such that for all $x, y \in \mathbb{R}^d_+$,

$$\begin{aligned} C^{-1}\left(\frac{x_d}{|x-y|} \wedge 1\right)^{\alpha-1}\left(\frac{y_d}{|x-y|} \wedge 1\right)^{\alpha-1}\frac{1}{|x-y|^{d-\alpha}} &\le G^0(x,y) \\ \le \quad C\left(\frac{x_d}{|x-y|} \wedge 1\right)^{\alpha-1}\left(\frac{y_d}{|x-y|} \wedge 1\right)^{\alpha-1}\frac{1}{|x-y|^{d-\alpha}} &. \end{aligned}$$

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$$\leq \quad C\left(\frac{x_d}{|x-y|} \wedge 1\right)^{\alpha-1}\left(\frac{y_d}{|x-y|} \wedge 1\right)^{\alpha-1}\frac{1}{|x-y|^{d-\alpha}}.$$

Note that when $\kappa = 0$ (no killing), regardless of the blow-up rate of the function $\mathcal{B}$, the decay rate of harmonic functions is given by $p = \alpha - 1$.

This should be compared with the BHP in (Bogdan-Burdzy-Chen, PTRF **127** (2003)) where they proved that the decay rate is $\alpha - 1$ when $\mathcal{B}$ is a positive constant.

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[KSV1], [KSV2] and [KSV3] deal with the elliptic theory. Can we establish the corresponding parabolic theory? That is, can we prove sharp two-sided heat kernel estimates? This is the topic of [CKSV].

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In [CKSV], we can actually also get heat kernel estimates for the corresponding reflected process. We state the result for the reflected case first.

Define

$$\mathcal{E}^0(u,v) = \frac{1}{2}\int_{\mathbb{R}^d_+}\int_{\mathbb{R}^d_+}(u(x)-u(y))(v(x)-v(y))J(x,y)\,dy\,dx.$$

Let $\overline{\mathcal{F}}$ be the closure of $C_c^\infty(\overline{\mathbb{R}^d_+})$ under $\mathcal{E}^0_1$. Then $(\mathcal{E}^0, \overline{\mathcal{F}})$ is a regular Dirichlet form on $L^2(\overline{\mathbb{R}^d_+})$. We denote the associated Hunt process by $\overline{Y}$ (reflected process)

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The first main result of [CKSV] is the following

Theorem 1

Suppose that **(A1)**, **(A3)** and **(A4)** hold. Then the process $\overline{Y}$ can be refined to be a conservative Feller process with strong Feller property starting from every point in $\overline{\mathbb{R}}^d_+$ and has a jointly continuous heat kernel $\bar{p} : (0, \infty) \times \overline{\mathbb{R}}^d_+ \times \overline{\mathbb{R}}^d_+ \to (0, \infty)$. Moreover, the heat kernel $\bar{p}$ has the following estimates: (a) When $d = 1$, for all $(t, x, y) \in (0, \infty) \times \overline{\mathbb{R}}^d_+ \times \overline{\mathbb{R}}^d_+$,

$$\bar{p}(t, x, y) \asymp t^{-d/\alpha} \wedge \Big(tJ(x + t^{1/\alpha}\mathbf{e}_d, y + t^{1/\alpha}\mathbf{e}_d)\Big).$$

(b) (i) When $d \geq 2$ and $\beta_2 < \alpha + \beta_1$, for all $(t, x, y) \in (0, \infty) \times \overline{\mathbb{R}}^d_+ \times \overline{\mathbb{R}}^d_+$,

$$\begin{aligned}\bar{p}(t, x, y) &\asymp \left(t^{-d/\alpha} \wedge \frac{tB_{\beta_1,\beta_2,\beta_3,\beta_4}(x + t^{1/\alpha}\mathbf{e}_d, y + t^{1/\alpha}\mathbf{e}_d)}{|x-y|^{d+\alpha}}\right)\\ &\asymp \left(t^{-d/\alpha} \wedge \frac{t}{|x-y|^{d+\alpha}}\right) B_{\beta_1,\beta_2,\beta_3,\beta_4}(x + t^{1/\alpha}\mathbf{e}_d, y + t^{1/\alpha}\mathbf{e}_d).\end{aligned} \tag{2}$$

Theorem 1 (Cont)

(b) (ii) If $\beta_2 > \alpha + \beta_1$, then for all $(t, x, y) \in (0, \infty) \times \overline{\mathbb{R}}^d_+ \times \overline{\mathbb{R}}^d_+$,

$$\bar{p}(t,x,y) \asymp \left(t^{-d/\alpha} \wedge \frac{t}{|x-y|^{d+\alpha}}\right) \Big[B_{\beta_1,\beta_2,\beta_3,\beta_4}(x + t^{1/\alpha}\mathbf{e}_d, y + t^{1/\alpha}\mathbf{e}_d) + \left(1 \wedge \frac{t}{|x-y|^{\alpha}}\right) B_{\beta_1,\beta_1,0,\beta_3}(x + t^{1/\alpha}\mathbf{e}_d, y + t^{1/\alpha}\mathbf{e}_d) \times \log^{\beta_3}\left(e + \frac{|x-y|}{((x_d \wedge y_d) + t^{1/\alpha}) \wedge |x-y|}\right) \Big]. \quad (3)$$

Theorem 1 (Cont)

(b) (iii) When $\beta_2 = \alpha + \beta_1$, then for all $(t, x, y) \in (0, \infty) \times \overline{\mathbb{R}}^d_+ \times \overline{\mathbb{R}}^d_+$,

$$\bar{p}(t, x, y) \asymp \left(t^{-d/\alpha} \wedge \frac{t}{|x-y|^{d+\alpha}} \right) \Big[B_{\beta_1, \beta_2, \beta_3, \beta_4}(x + t^{1/\alpha}\mathbf{e}_d, y + t^{1/\alpha}\mathbf{e}_d)$$

$$+ \left(1 \wedge \frac{t}{|x-y|^{\alpha}} \right) B_{\beta_1, \beta_1, 0, \beta_3+\beta_4+1}(x + t^{1/\alpha}\mathbf{e}_d, y + t^{1/\alpha}\mathbf{e}_d)$$

$$\times \log^{\beta_3}\left(e + \frac{|x-y|}{((x_d \wedge y_d) + t^{1/\alpha}) \wedge |x-y|} \right) \Big]. \qquad (4)$$

Now we state sharp two-sided heat kernel estimates for Y. Fix $\kappa \in [0, \infty)$. That is, we are dealing with the case either with or without critical killing.

Theorem 2 [CKSV]

Suppose that **(A1)–(A4)** and $q \in [(\alpha - 1)_+, \alpha + \beta_1)$. Then the process Y^κ can be refined to start from every point in $\mathbb{R}^d_+$ and has a jointly continuous heat kernel $p^\kappa : (0, \infty) \times \mathbb{R}^d_+ \times \mathbb{R}^d_+ \to (0, \infty)$. Moreover, the following approximate factorization holds for all $(t, x, y) \in (0, \infty) \times \mathbb{R}^d_+ \times \mathbb{R}^d_+$:

$$\begin{aligned} p^\kappa(t, x, y) &\asymp \left(1 \wedge \frac{x_d}{t^{1/\alpha}}\right)^q \left(1 \wedge \frac{y_d}{t^{1/\alpha}}\right)^q \bar{p}(t, x, y) \\ &\asymp \mathbb{P}_x(\zeta^\kappa > t)\mathbb{P}_y(\zeta^\kappa > t)\, \bar{p}(t, x, y), \end{aligned} \tag{5}$$

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Integrating our heat kernel estimates, we can get

Theorem 3

Suppose that **(A1)**, **(A3)** and **(A4)** hold. If $d > \alpha$, then

$$\bar{G}(x,y) \asymp \frac{1}{|x-y|^{d-\alpha}}, \quad x,y \in \overline{\mathbb{R}}^d_+. \tag{6}$$

If $d \le \alpha$, then $\bar{G}(x,y) = \infty$ for all $x,y \in \overline{\mathbb{R}}^d_+$.

Define $H_q(x,y$ by

$$\begin{cases} 1 & \text{if } q < \alpha + \frac{1}{2}(\beta_1+\beta_2), \\ \log^{\beta_4+1}\left(e + \frac{|x-y|}{(x_d \vee y_d)\wedge |x-y|}\right) & \text{if } q = \alpha + \frac{1}{2}(\beta_1+\beta_2), \\ \left(\frac{x_d \vee y_d}{|x-y|}\wedge 1\right)^{2\alpha+\beta_1+\beta_2-2q} \log^{\beta_4}\left(e + \frac{|x-y|}{(x_d \vee y_d)\wedge |x-y|}\right) & \text{if } q > \alpha + \frac{1}{2}(\beta_1+\beta_2). \end{cases}$$

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Theorem 4

Suppose that **(A1)–(A4)** and $q \in [(\alpha-1)_+, \alpha+\beta_1)$. When $\alpha \le 1$, suppose also that $q > 0$ (or, equivalently, $\kappa > 0$). Then G^κ has the following estimates:

(i) If $d \ge 2$, then for all $x, y \in \mathbb{R}^d_+$,

$$G^\kappa(x,y) \asymp \frac{H_{q_\kappa}(x,y)}{|x-y|^{d-\alpha}} \left(\frac{x_d \wedge y_d}{|x-y|} \wedge 1\right)^q \left(\frac{x_d \vee y_d}{|x-y|} \wedge 1\right)^q .$$

(ii) If $d = 1$, then for all $x, y \in \mathbb{R}^d_+$,

$$G^\kappa(x,y) \asymp \begin{cases} \dfrac{1}{|x-y|^{1-\alpha}} \left(\dfrac{x \wedge y}{|x-y|} \wedge 1\right)^q & \text{if } \alpha < 1, \\ \left(\dfrac{x \wedge y}{|x-y|} \wedge 1\right)^q \log\left(e + \dfrac{(x\wedge y) \vee |x-y|}{|x-y|}\right) & \text{if } \alpha = 1, \\ (x \wedge y)^{\alpha-1} \left(\dfrac{x \wedge y}{|x-y|} \wedge 1\right)^{q-\alpha+1} & \text{if } \alpha > 1. \end{cases}$$

Thank you!